Napkin Notes on Algebra

The origin of algebra as a formal study is a book written by an Arab named Al-kwarizmi, written in about the early to mid-800's. One of the words in the title is "al-jabr". In 1120 it was translated in English and the "al-jabr" in the title of the original book was translated as "algebra" in the title of the translation. This is where the word "algebra" started in Europe (in the Latin language).

"Al-jabr", means something like "completion" or "restoring" or "reunion" in Arabic and it was used in the original book to describe what happens when you remove a subtracted quantity from one side of the equation and add it back to the other side. Example: If 30 = X - 5, then you can add 5 to both sides (you perform an "al-jabr" to get 35=X.

Now think of algebra as the art of keeping the two sides of an equations "balanced" or equal. The best way of introducing algebra I have found is to talk about the old balancing scales. If you have a box on the left side of the scale with an unknown number of marbles in it, and there are three loose marbles outside of the box on the left side of the scale, and you also have 8 marbles on the right side of the scale. Then how many marbles are in the box on the left?

The equation that shows this is: X + 3 = 8. Think of the situation in the previous paragraph. To determine how many marbles are in the box, you can just do a trial and error guess, until you realize that 5 + 3 = 8, but if you want a method, what you do is remove the three loose marbles on the left and three of the eight on the right. That keeps the scale balanced and you now have only the box on the left and five marbles on the right, so obviously there must be five in the box on the left (If you want to be picky, you'll have to say that the weight of the box itself doesn't count, or that the box is very, very light and it doesn't affect the balancing.)

So now, what if we had two sealed boxes and four loose marbles on the left, and ten marbles on the right side of the scales? What equation does that represent, and how do you solve it?

The illustration doesn't work too well for equations that have a subtraction, like X - 2 = 7. You can rig it up by saying that there is a box on the left, and also there are two helium balloons attached to the scale on the left with each helium balloon having the pulling-up strength equal to the pushing down strength of one marble. (And there are seven marbles on the right). But this might be too much for the beginner. Save it for later. If you cut loose the string attaching the two balloons on the left, you have to *add two marbles* from the right two marbles to the right to keep it balanced. (Save this for a later lesson.)

Explain that when you are solving an equation, you have to "get rid of" everything except the variable on one side (which is "solving" for the variable). So in X + 2 = 6. You get rid of the plus two by subtracting two from both sides. Emphasize that in order to get rid of something, you have to *do* the *opposite* of what put it there. If you start at the 5th square of a sidewalk and jump forward two squares, that puts you on the 7th square. If you jump back two squares, you just "did the opposite" and are back where you started. This shows that:

5 + 2 = 7 and 5 + 2 - 2 = 5. The "+ 2" and the "- 2" in the second one "cancel each other out" (you can draw slashes through them).

So in algebra: an unknown quantity plus two minus two, just leaves you with the unknown quantity. This "doing the opposite" is how you "get rid of" things on one side of an equation:

X + 3 = 10

X + 3 - 3 = 10 - 3   (you have to subtract the same amount from both sides)

The "+ 3" and "- 3" on the left cancel each other out, so you have:

X = 10 - 3

then simplify the right side, to get:

X = 7

Same thing with multiplying, if I start with 5 things and triple it, I get 15 things. If I "un-triple" the 15, I am back at 5 where I started:

5 X 3 = 15

5 X 3 [division symbol] 3 = 15 [division symbol] 3,

draw a slash through the "X 3" and another slash through the "[division symbol] 3"

simplify the right side, then:

5 = 5

Replace the starting 5 above with an X.

At this point you should be using a centered dot to show multiplication instead of an "X" which gets confused with the variable X.

Explain that the division symbol (horizontal line with a dot above and below) stands for division just like a fraction line does. (Never use a slanted fraction line, always use a horizontal one.)

So:

X [division symbol] 3 = 4

then,

X [division symbol] 3 [dot] 3 = 4 [dot] 3 (triple each side to undo the untripling)

then cancel,

X = 4 [dot] 3,

then simplify,

X = 12

All of this can be done like this:

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  X
------  =  4
  3
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  X
-----  [dot] 3  = 4 [dot] 3,
  3
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draw slashes through the 3 in the denominator and the

[dot] 3 next to it. Etcetera....

Remember when you solve equations by "doing opposites" (subtracting is the opposite of adding, and adding is the opposite of subtracting. Dividing is the opposite of multiplying and multiplying is the opposite of dividing) what you are doing is getting rid of the weakest link first.

Example,

X [dot] 3 + 2 = 11

The + is the "weakest link" on the left, because it is *a lower priority* on the Please-excuse-my-dear-Aunt-Sally scale than multiplication is!

***** Solving equations is like peeling an onion, starting with the outer layers *******

***** This means you are going in reverse order strength in P, E, MD, AS ******

BE CAREFUL to not let parentheses trip things up. Example:

(X + 5) [dot] 3 = 24

The parentheses makes the plus symbol inside "a stronger link" than the multiplication on the left!!! (Because of the normal P, E, MD, AS order-of-operations list.)

So therefore you must get rid of (undo) the [dot] three first by dividing both sides by 3. Also remember that the fraction line is a "grouping symbol" which means that whenever you use it there are invisible parentheses. Example:

$$\frac{X + 5}{3} = 7$$

Is the same as:

$$\frac{(X + 5)}{3} = 7$$

This shows that the plus is "made stronger" because of the parentheses, so you have to get rid of the divide by three first by multiplying both sides by 3. THEN you subtract five from both sides.

From here you can start making the multiplication "dot" invisible, too, so that:

3 [dot] X + 5 becomes 3X + 5 (X is a variable, of course)

and you can solve stuff like:

3X + 5 = 20

Plus is a weaker link, so subtract 5 from both sides, the divide both sides by 3.

Wednesday, December 25, 2002.

A Tip on Percentages

Remember what "per cent" means--literally, "Per one-hundredth part."

So when you are talking about percent increases, you are asking:

If I broke the original number into 100 parts, how many extra parts (of that size) would I be adding with this increase?

So let's say your original amount was 300---

Break that into 100 parts to see that a "hundredth part" is 3 units.

Now if our new amount (after the increase) was 345---that means we added 45 units, which is 15 extra "hundredth parts"---meaning that it is a 15% increase.

Wednesday, March 26, 2003

Positive and Negative Numbers

Consider:

7 - 4 ("Take four away from seven"). It leaves you the same result as 7 + -4 ("Add a debt of four to seven" or "Add a taking away of four to seven"). Now consider the three debts of four dollars, as indicated in this expression: 7 + -4 + -4 + -4, which means the same as "add three four-dollar debts to your original balance of seven dollars." Another way to write that is 7 + 3(-4).

Now let's suppose you had 40 dollars in your pocket and you also had three five-dollar debts. Your net worth would be: 40 + 3(-5) = 25 dollars. Now starting with your net worth of 25 dollars, let's remove one of the five-dollar debts. Since "removing" means "subtract a debt," we can calculate our new net worth as being 25 - (-5) = 30 dollars.

Let's turn a verb ("remove") into its noun form ("removal") by adding "a removal" of a debt of five dollars to 25. In math language, that is: 25 + -(-5) = 30 dollars. Another way to look at it is "adding *one* removal," i.e., 25 + (-1)(-5) = 30.If we remove all three five-dollar debts from 25, we would have 25 - 3(-5) = 40 dollars. Changing the verb "remove" to the noun "removal" and performing a rephrasing gives us "adding three removals a five-dollar debt", we have: 25 + (-3)(-5) = 40 dollars.

So we see that "(-3)(-5)" ("three removals of a five-dollar debt" or "three removals of a subtracting of five-dollars") results in the addition of 15 dollars to our net worth, and so therefore, a negative integer times a negative integer produces a positive integer.
I'm claiming that positive integers are noun forms of the verb "to add." So, e.g., 25 + +15 ("25 plus positive 15" or "Add an adding of 15 to 25") results in the same thing as 25 + 15 ("Add 15 to 25").

So, positive 15 (the integer) is not the same thing as plain 15 (the natural number), because "positive 15" means "the adding of 15" and "plain 15" just means 15. The two ideas are different, just as the phrase "drinks water" is not the same thing as the word "water," even though both ideas involve the concept water. "Plain 15" refers simply to 15 things, just like a collective noun. "Positive 15" refers to an action being taken, and that action is expressed in a noun form, just as "the drinking of" is a noun form of "drink."

What confuses people who try to make sense out of why a negative integer times a negative integer equals a positive integer is that they think that a plain number is interchangeable with its corresponding integer. It is not. Then they scratch their heads, wondering why the multiplication of two "negative quantities" (things that seemingly don't exist) gives a plain number, which refers to something that does exist. It's as if two non-things multiply to give a thing. This is the same thing as confusing "drinks water" with "water." The two are highly similar and related, but they are not identical or interchangeable.

Of course, -2, for example, does not represent two "non-things." It means "the taking away of 2 things" or "the leftward moving of two units" on a number line. In the latter case, we see how the concepts of "positive" and "negative" refer to the idea of direction. Integers then, can be seen as being one-dimensional vectors, an insight that is rarely articulated to by mathematicians (since probably very few even understand it). Even though the one-dimensional vector <-2> has only one component (in the way that the term "component" is normally used in math), that one component itself is made up of two components of a different kind, the negative component (the directionality) and the natural component (the "2"). Each integer is composed of a directionality-component and a natural-number component.

The only exception to this would be the integer zero, but that's a whole 'nother story. "Zero" is really just a figure of speech. It means the absence number. It's a concept of method, not a concept of entities (as the natural numbers are) and not a combination concept of aspects of entities/concept of entities (as integers are, with directionality sub-component being an *aspect* of entities and the natural number sub-component being a normal concept of entities).

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